

Čech cocycles for quantum principal bundles

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Abstract. In order to study connections and gauge theories on noncommutative spaces it is useful to use the local trivializations of principal bundles. In this note we show how to use noncommutative localization theory to describe a simple version of cocycle data for the bundles on noncommutative schemes with Hopf algebras in the role of the structure group which locally look like Hopf algebraic smash products. We also show how to use these Čech cocycles for associated vector bundles. We sketch briefly some examples related to quantum groups.

1. Introduction

It is often in noncommutative geometry that the Hopf algebras appear in the role of symmetry groups ([5, 9]). Thus one approach of developing the gauge theories on noncommutative spaces is to develop first the theory of principal bundles whose structure group is replaced by a Hopf algebra and then to develop the concept of connection on such a principal bundle, and finally to study the action functionals developed on this basis ([2]). In noncommutative geometry based on C^* -algebras, it is sufficient to model spaces with a single global algebra, while in the algebraic setting, a single algebra corresponds just to an affine algebraic variety. For a more general variety, one needs to glue local (affine) charts ([10, 12, 22]). One flavour of such gluing is the gluing of categories of modules via noncommutative localization ([12, 19, 21, 22]).

In the case of noncommutative affine varieties playing the role both of total and base space of a principal bundle, the generally accepted notion of a Hopf algebraic principal bundle is a Hopf-Galois extension [9, 11, 13] and its coalgebra generalizations ([3]); the trivial bundle on the other hand is the special case of Hopf algebraic smash product. In this affine case, there are sensible and much studied proposals ([2]) how to introduce connections, gauge transformations, curvature etc. What is not clear in the published literature is how to glue such data between local charts. Related problem is how to generalize these data to nonaffine spaces, and in particular of noncommutative schemes [12]. Thus we are interested in gluing data for nonaffine noncommutative principal bundles first and in the next step of gluing the sections of associated vector bundles. The connection forms may be treated as a special kind of such sections, in the presence of a well behaved noncommutative differential calculus.

In an ongoing project with Gabi Böhm, we study a global generalization of Hopf-Galois extensions and also the corresponding cocycle data at the categorical level: the cocycles there are made out of functors ([1]). That is a clean general approach, but it is difficult to use it directly. In this note I will present how to work out directly a special case, where one can define and build cocycles at the level of homomorphisms of algebras, rather than functors. This direct approach will be likely easier to use in physical situations.

2. Equivalences of smash products

The basic ingredient of our construction of cocycles comes from the analysis of equivalences of Hopf smash products. Even more general case of equivalences of cocycled crossed products is fully characterized by Doi ([4]).

We use the Sweedler's notation for coproduct $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$, and its extension to right coactions $\rho(e) = \sum e_{(0)} \otimes e_{(1)}$ and left coactions $\rho(v) = \sum v_{(-1)} \otimes v_{(0)}$ ([9, 11]). Given an action \triangleright of a bialgebra H on an algebra U which is Hopf (i.e. makes it a H -module algebra: $h \triangleright (uv) = \sum (h_{(1)} \triangleright u) \otimes (h_{(2)} \triangleright v)$ and $h \triangleright 1 = \epsilon(h)1$, [9, 11]), the tensor product $U \otimes H$ (where \otimes for elements is traditionally written \sharp) has nontrivial multiplication $(u \sharp h)(v \sharp g) = \sum u(h_{(1)} \triangleright v) \sharp h_{(2)} g$ and a coaction $u \sharp h \mapsto \sum (u \sharp h_{(1)}) \otimes h_{(2)}$, which together form a structure of a right H -comodule algebra $U \sharp H$ which will be referred to as the smash product. If H is a Hopf algebra with antipode S (as we assume from now on) then $\gamma \circ S$ is the inverse γ^{-1} of γ in the space of linear maps $Hom(H, E)$ with respect to the convolution product $(f_1 \star f_2)(h) = \sum f_1(h_{(1)}) \cdot_E f_2(h_{(2)})$. A trivial principal H -bundle over U will be a left U -module, right H -comodule algebra which is isomorphic to $U \sharp H$ as U -module and H -comodule algebra. Morphisms of trivial bundles will preserve these structures. If E is a left U -module right H -comodule, then the existence of the isomorphism $\xi : U \sharp H \rightarrow E$ is equivalent to the existence of a morphism of right H -comodule algebras $\gamma : H \rightarrow E$, namely $\gamma(h) = \xi(1 \sharp h)$, which we call the trivializing section. The appropriate action is then $h \triangleright a = \sum \gamma(h_{(1)}) u \gamma(S h_{(2)})$ where $h \in H$ and $u \in U$.

Let now $f : E_1 \rightarrow E_2$ be a morphism of bundles over U where $\gamma_i : H \rightarrow E_i$, $i = 1, 2$ are given trivializations. Then $f \circ \gamma_2$ is also a trivializing section of E_2 . Thus for comparing the trivial bundles it is enough to compare different trivializing sections of the same bundle when $f = 1$ and $E_1 = E_2 = E$. A generic element in E can be written as $\sum_k u_k \gamma_2(h_k)$. The element $\gamma_1(h)$ is of the form $\sum y(h_{(1)}) \gamma_2(h_{(2)})$ for some algebra map $y : H \rightarrow E$. Indeed, y can be obtained as $(1 \otimes \epsilon) \xi_2^{-1}(\gamma_1(h))$ where $\xi^{-1} : E \rightarrow U \sharp H$ is the isomorphism induced by the section γ_2 . Thus

$$y(h) = \sum \gamma_1(h_{(1)}) \gamma_2^{-1}(h_{(2)}) \quad (1)$$

encodes all the information on comparing different trivializations. Furthermore,

$$\begin{aligned} \sum (h_{(1)} \triangleright_1 u) y(h_{(2)}) &= \sum (h_{(1)} \triangleright_1 u) \gamma_1(h_{(2)}) \gamma_2^{-1}(h_{(3)}) \\ &= \sum y(h_{(1)}) \gamma_2(h_{(2)}) u \gamma_1(S h_{(3)}) \gamma_1(h_{(4)}) \gamma_2^{-1}(h_{(5)}) = \sum y(h_{(1)}) (h_{(2)} \triangleright_2 u). \end{aligned}$$

Therefore

$$\sum (h_{(1)} \triangleright_1 u) y(h_{(2)}) = \sum y(h_{(1)}) (h_{(2)} \triangleright_2 u) \quad (2)$$

Furthermore it is easy to see that a composition of two morphisms of trivial bundles corresponds to a convolution product of the corresponding y -maps:

$$y_{13}(h) = \sum \gamma_1(h_{(1)}) \gamma_3^{-1}(h_{(2)}) = \sum \gamma_1(h_{(1)}) \gamma_2^{-1}(h_{(2)}) \gamma_2(h_{(3)}) \gamma_3(h_{(4)}) = \sum y_{12}(h_{(1)}) y_{23}(h_{(2)})$$

3. Čech cocycle

3.1. Noncommutative localization and noncommutative schemes

We start with rather technical general sketch and then we give a simple recipe of a special case of our interest. A noncommutative space in algebraic sense is ultimately represented by an abelian category \mathcal{A} whose objects are viewed as quasicoherent sheaves. The category is covered by *flat localization* functors $Q_\lambda^* : \mathcal{A} \rightarrow \mathcal{A}_\lambda$, $\lambda \in \Lambda$ having right adjoint functors $Q_{\lambda*}$, and \mathcal{A}_λ is isomorphic to the category of left modules over a noncommutative algebra U_λ ([19, 12]). Denote also $Q_\lambda = Q_{\lambda*} Q_\lambda^* : \mathcal{A} \rightarrow \mathcal{A}$. Thus we may be given a family of algebras U_λ , $\lambda \in \Lambda$ viewed as algebras of functions on Zariski open charts whose *intersections* will be replaced by considering mixed iterates $Q_\lambda^* Q_{\mu*} Q_{\mu*}^* : \mathcal{A} \rightarrow \mathcal{A}_{\mu\lambda} \subset \mathcal{A}_\lambda$ of localization functors where $\mathcal{A}_{\mu\lambda}$ is the essential image of the written functor with values in \mathcal{A}_λ . The bad thing is that $\mathcal{A}_{\mu\lambda}$ is not necessarily equivalent to $\mathcal{A}_{\lambda\mu}$, nor $Q_\lambda Q_\mu \cong Q_\mu Q_\lambda$, and worse, with A_λ looking affine, i.e. like the category of modules over and algebra, the iterates $\mathcal{A}_{\lambda\mu}$ do not look like that in general, hence we can *not* talk about the *algebra* $U_{\lambda\mu}$, but rather only a bimodule.

3.2. Mixed localizations and gluing bimodules

The double consecutive localization $\mathcal{A}_{\lambda\mu} \subset \mathcal{A}_\mu$, hence $\mathcal{A}_{\lambda\mu}$ is always a subcategory of the category of all left U_μ -modules. One can still define a left U_μ -module $U_{\lambda\mu} := Q_\mu(U_\lambda)$. Under mild conditions, it is a U_μ - U_λ -bimodule flat from both sides; this bimodule is generated by the image of unit element in U_λ under the adjunction map $U_\mu \rightarrow U_{\lambda\mu}$. This will be the working assumption in this article. In fact, if the family of localizations is finite, this can be abstracted further to the noncommutative space covers of Kontsevich-Rosenberg ([8]) which are given in terms of a coring with an additional "structure map"; it seems that the whole theory of this article could be generalized to their setup (see [15] for some hints) In the simplest affine case, when there is a global algebra U such that $U_\lambda = Q_\lambda(U)$ for all λ , the localization functor Q_μ is isomorphic to $U_\mu \otimes_U$ and $U_{\lambda\mu} = U_\mu \otimes_U U_\lambda$. Notice also that we can consider higher iterates like $U_{\lambda\mu\nu} = U_\nu \otimes_U U_\mu \otimes_U U_\lambda$ and various natural maps from lower into higher iterates. We are more interested in the case when $U_\lambda = E_\lambda^{\text{co}H}$ is the subalgebra of coinvariants in a right H -comodule algebra E_λ , i.e. the elements $u \in E_\lambda$ for which the coaction is of the form $\rho(u) = u \otimes 1$; and such that $E_\lambda = Q_\lambda(E)$ is the localized algebra of some algebra E . We say that U_λ is the algebra of localized H -coinvariants in E . In good cases such

algebras can be glued to form a noncommutative *quotient space* of E , the method which we pioneered in [18]. Now we can define $U_{\lambda\mu} = E_{\lambda\mu}^{\text{co}H}$ and similarly for higher iterates. This will be important for Čech cocycles.

3.3. Locally trivial bundles

Suppose we are now given system of algebras $\{U_\lambda\}_{\lambda \in \Lambda}$ as above. We should technically require that their categories of modules glue appropriately along $U_{\lambda\mu}$ and $U_{\mu\lambda}$ via flat descent to the global category \mathcal{A} . As this is treated at length elsewhere we just suppose that the standard setup, e.g. of noncommutative schemes is understood; we also follow the assumptions from the previous section about double iterates. While one can define a rather abstract notion of principal H -bundle on \mathcal{A} ([1]) we will not do that here (but the comparison is due in a sequel paper). Instead we propose the new explicit concept of **noncommutative Čech 1-cocycle with coefficients in the Hopf algebra H** :

- (1) Each U_λ is equipped with a Hopf action $H \triangleright_\lambda U \rightarrow U$
- (2) For each ordered pair λ, μ there is a map $y_{\mu\lambda} : H \rightarrow U_{\lambda\mu}$ (notice the order of labels and that the codomain is not generally an algebra but a bimodule) such that

$$\sum (h_{(1)} \triangleright_\mu u) y_{\mu\lambda}(h_{(2)}) = \sum y_{\mu\lambda}(h_{(1)}) (h_{(2)} \triangleright_\lambda u) \quad (3)$$

Notice that the multiplications on the left and right are due the bimodule structure on $U_{\lambda\mu}$.

- (3) For each ordered triple λ, μ, ν we have the cocycle conditions: for each $h \in H$,

$$y_{\nu\lambda}(h) = \sum y_{\nu\mu}(h_{(1)}) y_{\mu\lambda}(h_{(2)})$$

in U_ν - U_λ -bimodule $U_{\lambda\mu\nu}$ and

$$y_{\mu\mu}(h) = 1$$

in $U_{\mu\mu} \cong U_\mu$.

Consider another set of such data with the same cover $\{U_\lambda\}_{\lambda \in \Lambda}$ (gluing data for the base space understood), but different H -actions $\tilde{\triangleright}_\lambda$ and different bundle transition maps $\tilde{y}_{\lambda\mu}$. The two sets are cohomologous if there is a 0-cocycle which is a family of linear maps $r_\lambda : H \rightarrow U_\lambda$ and this 0-cocycle relates the above 1-cocycles as follows:

- (1) each r_λ is convolution invertible
- (2) r_λ exhibits the equivalence of \triangleright_λ actions $\triangleright_\lambda, \tilde{\triangleright}_\lambda$:

$$\sum (h_{(1)} \triangleright_\lambda u) r_\lambda(h_{(2)}) = \sum r_\lambda(h_{(1)}) (h_{(2)} \tilde{\triangleright}_\lambda u) \quad (4)$$

- (3) for each ordered pair (μ, λ) ,

$$\sum r_\lambda(h_{(1)}) y_{\lambda\mu}(h_{(2)}) = \sum y_{\lambda\mu}(h_{(1)}) r_\mu(h_{(2)}) \quad (5)$$

holds in $U_{\mu\lambda}$.

4. Examples over quantum flag varieties

Given an indeterminate q , $M_q(2)$ is the associative $\mathbf{C}[q, q^{-1}]$ -algebra with generators $a = t_1^1, b = t_2^1, c = t_1^2, d = t_2^2$ subject to the relations $ab = qba, ac = qca, bd = qdb, cd = qdc, bc = cb$ and $ad - da = (q - q^{-1})bc$. It generalizes to $M_q(n)$ which is generated by t_j^i where $i, j = 1, \dots, n$ and for every pair $i < j, k < l$ of labels modulo any set of relations such that setting $a = t_k^i, b = t_l^i, c = t_k^j, d = t_l^j$ generates the subalgebra which is a copy of $M_q(2)$. It is convenient to form the matrix $T = (t_j^i)$. The algebra $GL_q(n)$ is the localization of $M_q(n)$ at the central element, the quantum determinant $D = \sum_{\sigma \in \Sigma} (-q)^{l(\sigma)} t_{\sigma(1)}^1 t_{\sigma(2)}^2 \cdots t_{\sigma(n)}^n \in M_q(n)$ where l is the length of the permutation σ . Similarly one defines quantum minors for the submatrices of T .

The formulas $\Delta(t_j^i) = \sum_{k=1}^n t_k^i \otimes t_j^k, \epsilon(t_j^i) = \delta_j^i$ uniquely extend to homomorphisms of algebras making $GL_q(n)$ a Hopf algebra with an antipode S such that $ST = T^{-1}$. The subalgebra generated by all t_j^i with $i < j$, is a Hopf ideal I and the quotient Hopf algebra will be referred to as quantum Borel subgroup B_q with generators $h_j^i = t_j^i + I$ and the projection map $\pi : GL_q(n) \rightarrow B_q(n)$ is given by $t_j^i \mapsto h_j^i$. B_q -coaction $\rho = (\text{id} \otimes \pi) \circ \Delta : GL_q(n) \rightarrow GL_q(n) \otimes B_q$ makes $GL_q(n)$ a right B_q -comodule algebra. We have earlier exhibited a family ([18]) of $n!$ Ore localizations of $GL_q(n)$ (Ore condition follows from [14]) which are ρ -compatible ([18, 21]) in the sense that the coaction extends to the localized algebra in a unique way making it a B_q -comodule algebra.

The construction of a coset space symbolically denoted by $GL_q(n)/B_q(n)$ (or its $SL_q(n)$ -version) is coming in a packet with the local trivialization: both stem from the geometric understanding of the quantum Gauss decomposition ([18, 17]). Namely, one decomposes the matrix of the generators T with rows permuted by the permutation matrix $w_\sigma^{-1}T$ as $U_\sigma A_\sigma$ where U_σ is an upper diagonal unidiagonal matrix and A_σ the lower triangular matrix, both with entries in the quotient skewfield of $GL_q(n)$. Here $\sigma \in \Sigma(n)$ is the element of the permutation group (the Weyl group for our case) and there are $n!$ such elements. The entries of U_σ and of A_σ together generate a subalgebra of the quotient field which is isomorphic to the Ore localization of $G = GL_q(n)$ by the set of principal (=lower right corner) quantum minors of $w_\sigma^{-1}T$. This localization G_σ is compatible with the coaction of quantum Borel so we have the induced coaction ρ_σ which makes G_σ a right $B_q(n)$ -comodule algebra. The entries of U_σ are coinvariant hence form a chart on the quantum homogeneous space (for more precise statement see [18, 17]). On the other hand, $\gamma_\sigma : B_q(n) \rightarrow G_\sigma$ is defined by the simple rule on generators

$$\gamma_\sigma(h_j^i) = (A_\sigma)_j^i$$

and extended as a homomorphism of algebras. To find the γ_σ hence it suffices to know how to do the Gauss decomposition of matrices with noncommutative entries and for the convolution inverse, one needs in addition to compute the antipode. Both problems can be done in terms of quasideterminants of Gel'fand and Retakh ([6, 19]), and the more detailed formulas are left for [17]. The quasideterminant involved, that is the quasiminors of T with some rows permuted, can further be expressed (up to a factor of

($-q$) to some power) as a ratio of two quantum determinants.

The Hopf action on the coinvariants is given on generators by

$$h_j^i \triangleright_\sigma (U_\sigma)_l^k = \sum_{i \geq s \geq j} \gamma_\sigma(h_s^i) (U_\sigma)_l^k \gamma_\sigma(Sh_j^s)$$

where γ_σ is a ratio of quasideterminants

$$\gamma_\sigma(h_s^i) = |T_{j,j+1,j+2,\dots,j+n}^{\sigma(i),\sigma(j+1),\sigma(j+2),\dots,\sigma(n)}|_{ij} |T_{j,j+1,j+2,\dots,j+n}^{\sigma(j),\sigma(j+1),\sigma(j+2),\dots,\sigma(n)}|_{jj}^{-1}$$

Thus we can easily find $y_{\sigma\tau}(h) = \sum \gamma_\sigma(h_1) \gamma_\tau(Sh_{(2)})$ on the generators. Notice that $y_{\sigma\tau}$ is not the homomorphism of algebras so one needs to go back to γ -s to compute it on an arbitrary given element.

For example on the simplest case of $G = SL_q(2)$ one has

$$\gamma_I(h_j^i) = \begin{pmatrix} a - bd^{-1}c & 0 \\ c & d \end{pmatrix}, \quad \gamma_\tau(h_j^i) = \begin{pmatrix} c - db^{-1}a & 0 \\ a & b \end{pmatrix}$$

where I is the trivial and τ the nontrivial permutation on two letters, while the matrix of transition maps is

$$Y = \begin{pmatrix} y(h_1^1) & y(h_2^1) \\ y(h_1^2) & y(h_2^2) \end{pmatrix} = \begin{pmatrix} -u & 0 \\ 1 & u' \end{pmatrix}, \quad Y^{-1} = \begin{pmatrix} -u' & 0 \\ u & 0 \end{pmatrix},$$

where $u = bd^{-1}$ is the generator in chart of the homogeneous space corresponding to the trivial permutation and $u' = db^{-1}$, the generators of the chart corresponding to the nontrivial permutation in $\Sigma(2)$. Though the base looks like \mathbf{CP}^1 at the local algebra level, its further structures are nonclassical (e.g. the measure utilized in [16]).

5. The associated vector bundles

For a right H -comodule algebra E and a left H -comodule V with coactions $\rho_E : E \rightarrow E \otimes H$ and $\rho_V : V \rightarrow H \otimes V$, the cotensor product is the vector subspace $E \square V \subset E \otimes V$ which equalizes $\rho_E \otimes \text{id}_V$ and $\text{id}_E \otimes \rho_V$. If E is a faithfully flat Hopf-Galois extension of $E^{\text{co}H}$ then E may be interpreted as a principal bundle and $E \square V$ as the space of global sections of the associated vector bundle with typical fiber V . We sketched in [16] that the cotensor products may be glued as well. The transition cocycles from the previous section may be used as well.

Let $v_j, j = 1, \dots, n$ be a basis of V (for simplicity, we consider the finite-dimensional fiber) and the coaction ρ_V in this basis is given by

$$\rho_V(v_i) = \sum_j v_i^j \otimes v_j$$

for some $v_i^j \in H$. The coaction axiom implies that $\Delta(v_i^j) = \sum_k v_i^k \otimes v_k^j$ (notice that the matrix multiplication is transposed). Consider now a cocycle for the principal bundle

with the notation from above. Then

$$\gamma_\lambda(v_i^j) = \sum_k y_{\lambda\mu}(v_i^k) \gamma_\mu(v_k^j)$$

Define the transition matrices $\mathfrak{M}_{\lambda\mu}$ by

$$(\mathfrak{M}_{\lambda\mu})_j^i = y_{\lambda\mu}(v_j^i) \in U_{\mu\lambda}.$$

Then $\mathfrak{M}_{\lambda\mu}^T \mathfrak{M}_{\mu\nu}^T = \mathfrak{M}_{\lambda\nu}^T$ and $\mathfrak{M}_{\lambda\lambda} = I$, where one interprets the results in appropriate iterated localizations and where $()^T$ is the sign for transposition (in $i \leftrightarrow j$).

Triviality of a Hopf-Galois extension $E^{\text{co}H} \hookrightarrow E$ implies triviality of the associated bundle in the following sense. If a right H -comodule E admits convolution invertible map of H -comodules $\gamma : H \rightarrow E$ then for any left H -comodule V there is an automorphism of left $E^{\text{co}H}$ -modules

$$\kappa_V^\gamma : E \otimes V \cong E \otimes V, \quad \kappa_V^\gamma(e \otimes v) = \sum e\gamma(v_{(-1)}) \otimes v_{(0)}$$

with inverse $\bar{\kappa}_V^\gamma : e \otimes v \mapsto \sum e\gamma^{-1}(v_{(-1)}) \otimes v_{(0)}$, and the automorphism κ_V^γ restricts to the isomorphism of $E^{\text{co}H}$ -modules

$$\kappa_V^\gamma| : E^{\text{co}H} \otimes V \rightarrow E \square V.$$

Now if the data $U_\lambda, \triangleright_\lambda, y_{\lambda\mu}$ form a cocycle of a principal H -bundle, then we can define the space of global sections of the associated bundle with typical fiber V as the vector subspace $\Gamma\xi_V$ of

$$\prod_{\lambda \in \Lambda} U_\lambda \otimes V$$

consisting of $|\Lambda|$ -tuples $(\sum_{i=1}^{n_\lambda} u_i^\lambda \otimes v_i)_\lambda$ such that $\sum_i u_i^\mu y_{\mu\lambda}(v_{i(-1)}) \otimes v_{i(0)} = \sum_j u_j^\lambda \otimes v_j$ in $U_{\lambda\mu} \otimes V$ for all ordered pairs (λ, μ) .

6. Conclusion: perspective toward gluing connections

As we know how to define the global sections of associated vector bundles, in particular we can do that for defining connections globally from local pieces. Available definitions of connections (see e.g. [2]) use as an ingredient the differential calculi over noncommutative algebras. The question of gluing the noncommutative calculi itself has some new elements, for example there is an additional condition of compatibility of a differential calculus with localizations involved in a cover. For that reason I have introduced in [19] the notion of *differential Ore condition* which might be useful.

Some nonaffine examples could already be constructed for homogeneous spaces of quantum groups. Some natural differential calculi there are known [7, 23] and we saw in this article how to define local trivializations for certain canonical principal bundles over quantum flag variety.

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